Just-in-Time preemptive scheduling around a common due date

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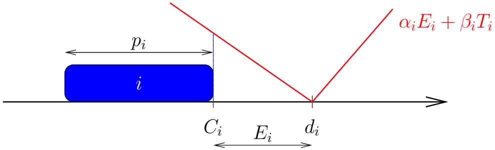
Preemption and JIT scheduling

- n operations (processing time p_i).
- **Preemption** is allowed.
- Find a **one-machine schedule** that minimize the total cost.
- How to define **job costs** to model the **Just-in-Time** philosophy?

Early-tardy completion

- Earliness-tardiness penalties $\alpha_i E_i + \beta_i T_i$



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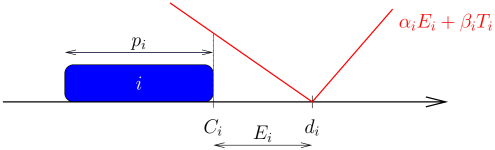
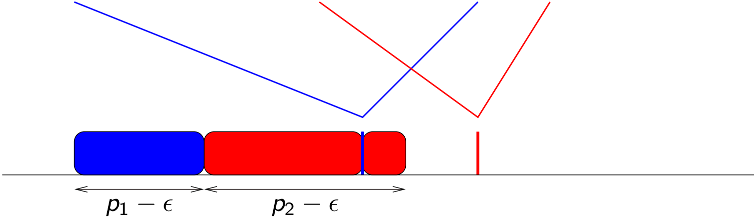


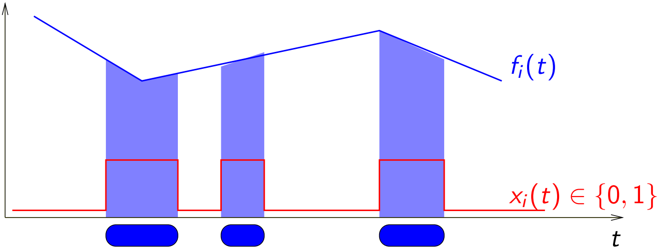
- But naive preemption...



- ...results in earliness relaxation

Position costs

- Cost functions $f_i : [0, \infty) \longrightarrow \mathbb{R}$
- Cost $f_i(t)dt$ when i is processed between t and $t + dt$.



- Cost of job i is

$$\int_0^\infty x_i(t) f_i(t) \mathrm{d}t$$

- $x_i(t)$ indicator function of the processing of job i.

$$\int_0^\infty x_i(t) \mathrm{d}t = p_i$$

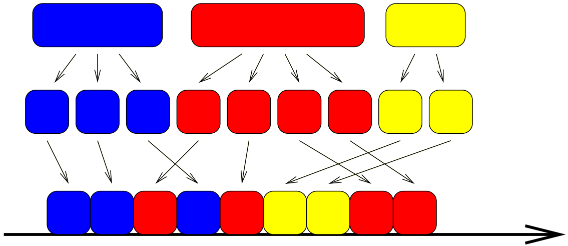
Objective function

- Minimize
$$\sum_{i=1}^{n} \int_0^\infty x_i(t) f_i(t) \mathrm{d}t$$
- Problem notation
$$1|\mathrm{pmtn}|\sum\int$$

Preemption at integer time points

[Sourd and Kedad-Sidhoum, JoS 2003]

- interruption only at **integer** time points
- tasks divided into **unit execution time** operations
- costs $c_{it} = \int_t^{t+1} f_i(t)\mathrm{d}t$ for scheduling a UET operation of job i in $[t, t+1)$



- but the size of the relaxed problem is **pseudopolynomial**

Dual problem

$$
\begin{array}{ll}
\text{min} & \sum_{i=1}^{n} \int_0^\infty x_i(t) f_i(t) \mathrm{d}t \\
\text{s.t.} & \int_0^\infty x_i(t) \mathrm{d}t = p_i \\
 & \sum_i x_i(t) \leq 1
\end{array}
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\text{s.t.} & \int_0^\infty x_i(t) \mathrm{d}t = p_i & \times u_i \\
 & \sum_i x_i(t) \leq 1 &
\end{array}
$$

Dual problem

$$L(u) = \min \quad \sum_{i=1}^{n} \int_0^\infty x_i(t)(f_i(t) - u_i)\mathrm{d}t + \sum_i u_i p_i$$

$$\text{s.t.}$$

$$\sum_i x_i(t) \leq 1$$

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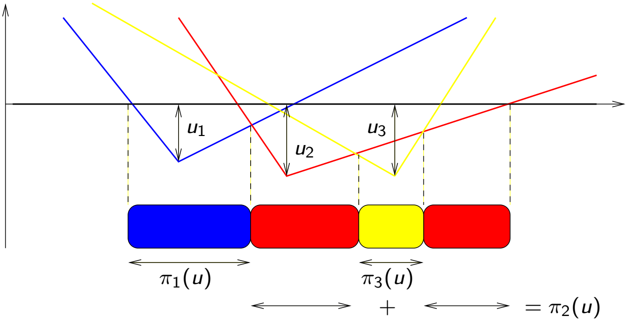
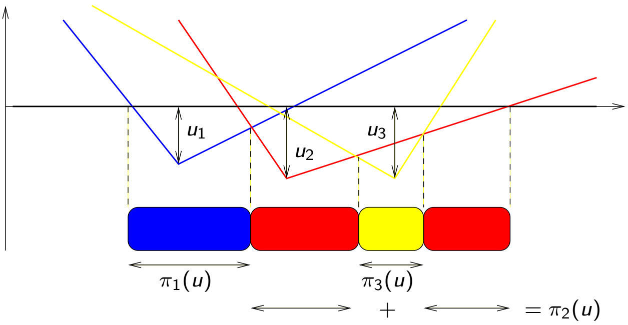


Dual problem

[Sourd, INFORMS JoC, 2004]



- **No duality gap**
- At the optimum,

$$(\pi_1(u), \pi_2(u), \cdots, \pi_n(u)) = (p_1, p_2, \cdots, p_n)$$

- **Polynomial** with the ellipsoid method

Motivation

- Study special easier cases
- Better understanding of this new criterion
- **Efficient strongly polynomial** algorithms

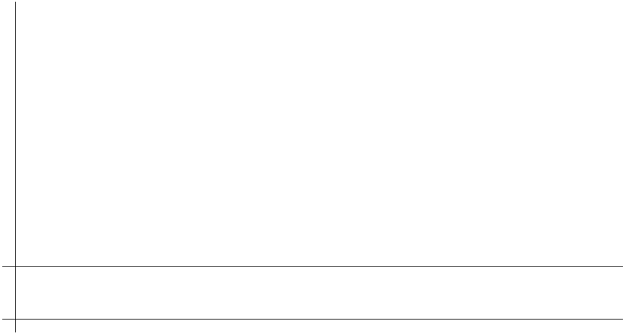
Today's problem

- Common due date d for each job
- Cost function

$$f_i(t) = \alpha_i \max(0, d - t) + \beta_i \max(0, t - d)$$

No earliness — $d = 0$

- $f_i(t) = \beta_i t$
- Larger slope first



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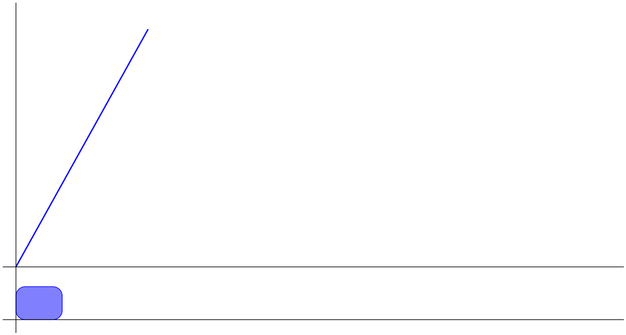

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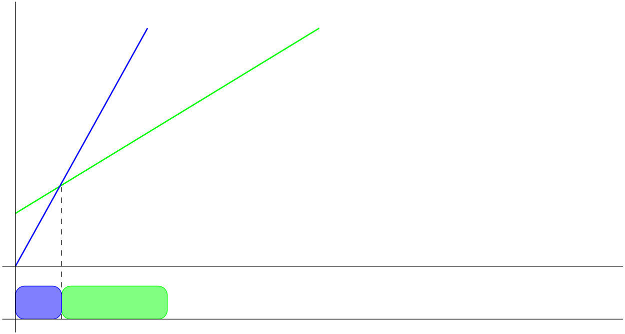

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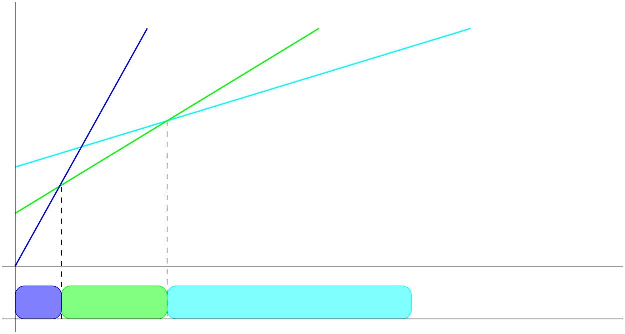

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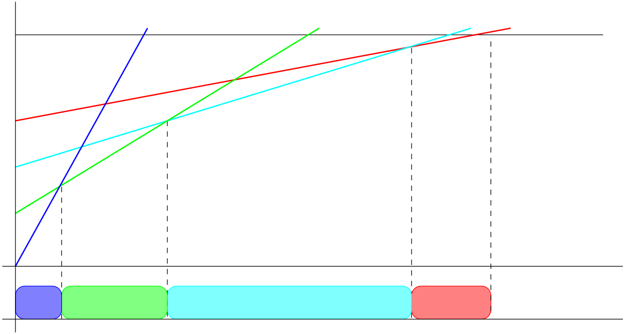


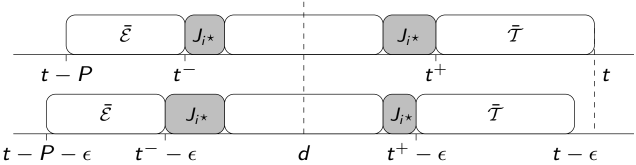
Basic properties of the solution

- An optimal schedule
  - starts at $t \leq d$
  - ends at $t + P \geq d$ with $P = \sum_i p_i$
  - no idle time in between the tasks
- the tardy parts of jobs are sorted according to the β_i
- the early parts of jobs are sorted according to the α_i

Rationale of the algorithm

- Let $f(t)$ be the optimal cost for scheduling all the jobs in $[t, t+P)$
- f is **convex**.
- Minimize the function f when t varies.
- Start with $t = d$ (jobs are all late).
- Compute $f(t-\epsilon)$ from $f(t)$ by maintaining the primal and dual solutions.
- End when the minimum of f is reached.

From $f(t)$ to $f(t-\epsilon)$



Lemma

Only **one** *job ($i^\star$) is* **transfered** *when t decreases.*

Sketch of the proof.

- The jobs in $\bar{\mathcal{E}}$ are **completely** early.
- The jobs in $\bar{\mathcal{T}}$ are **completely** tardy.
- the dual variables of the job in between $i^\star$ do not change

□

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- A primal approach computationally more efficient
- **Marginal transfer cost**
  - if job i is transfered

$$f(t-\epsilon) = f(t) + m_i\epsilon + o(\epsilon)$$

  - $m_i = \sum_j \min(\alpha_j, \alpha_i) p_j^- - \min(\beta_j, \beta_i) p_j^+$
- Select the job with the **smallest** marginal transfer cost.
- The variation of m_i is (piecewise) linear.

$$m_i(t-\epsilon) = m_i(t) + \left(\min(\alpha_i, \alpha_{i^\star}) + \min(\beta_i, \beta_{i^\star})\right)\epsilon$$

Events

- **Discretize** the "continuous" procedure
- Classes of **events**
  1. Transfer if job $i^\star$ completed
  2. Another job becomes critical
  3. $t = 0$
  4. Minimum of f is reached
- As the variation of the marginal costs are linear, the distance between the current event and the next event can be easily computed.

Number of events

Lemma

The transfer of a job can only be interrupted by a wholly late job.

Corollary

There are $O(n)$ events.

Complexity

Theorem

The algorihm runs in $O(n^2)$ time.

Proof.

- There are $O(n)$ events
- Marginal transfer costs are updated in $O(n)$ time.
- Next event is calculated in $O(n)$ time.

$\square$

Conclusion

- An $O(n^2)$ algorithm for the common due date problem
- Release dates, deadlines ?
- Non common due dates ??
- Lower bound for the non-preemptive problem.